

Karhunen–Loève expansion for a generalization of Wiener bridge

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Abstract

We derive a Karhunen–Loève expansion of the Gauss process $B_t - g(t) \int_0^1 g'(u) dB_u$, $t \in [0, 1]$, where $(B_t)_{t \in [0, 1]}$ is a standard Wiener process and $g : [0, 1] \rightarrow \mathbb{R}$ is a twice continuously differentiable function with $g(0) = 0$ and $\int_0^1 (g'(u))^2 du = 1$. We formulate two special cases with the function $g(t) = \frac{\sqrt{2}}{\pi} \sin(\pi t)$, $t \in [0, 1]$, and $g(t) = t$, $t \in [0, 1]$, respectively. The latter one corresponds to the Wiener bridge over $[0, 1]$ from 0 to 0.

1 Introduction

In this note we present a new class of Gauss processes, generalizing the Wiener bridge, for which Karhunen–Loève (KL) expansion can be given explicitly. We point out that there are only few Gauss processes for which the KL expansion is explicitly known. To give some examples, we mention the Wiener process (see, e.g., Ash and Gardner [3, Example 1.4.4]), the Ornstein–Uhlenbeck process (see, e.g., Papoulis [19, Problem 12.7] or Corlay and Pagès [7, Section 5.4 B]), the Wiener bridge (see, e.g., Deheuvels [8, Remark 1.1]), Kac–Kiefer–Wolfowitz process (see, Kac, Kiefer and Wolfowitz [11] and Nazarov and Petrova [18]), weighted Wiener processes and bridges (Deheuvels and Martynov [9]), Jandhyala–MacNeill process (Jandhyala and MacNeill [10, Section 4]), a generalization of Wiener bridge (Pycke [20]), generalized Anderson–Darling process (Pycke [21]), Rodríguez–Viollaz process (Pycke [22]), scaled Wiener bridges or also called α -Wiener bridges (Barczy and Iglói [4]), detrended Wiener processes (Ai, Li and Liu [2]), additive Wiener processes and bridges (Liu [13]), additive Slepian processes (Liu, Huang and Mao [14]) and Spartan spatial random fields (Tsantili and Hristopulos [23]). We also mention that KL expansions of Gauss processes have found several applications in small deviation theory, for a complete bibliography, see Lifshits [12]. Here we only mention two papers of Nazarov and Nikitin [15], [17].

Let \mathbb{Z}_+ , \mathbb{N} and \mathbb{R} denote the set of non-negative integers, positive integers and real numbers, respectively. For $s, t \in \mathbb{R}$, we will use the notation $s \wedge t := \min\{s, t\}$. Let $(B_t)_{t \in [0, 1]}$ be a standard Wiener process, and let $g : [0, 1] \rightarrow \mathbb{R}$ be a twice continuously differentiable function such that $g(0) = 0$ and $\int_0^1 (g'(u))^2 du = 1$. Let us introduce the process

$$(1.1) \quad Y_t := B_t - g(t) \int_0^1 g'(u) dB_u, \quad t \in [0, 1].$$

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One can consider $Y = (Y_t)_{t \in [0,1]}$ as a generalization of the Wiener bridge corresponding to the function g . In the special case $g(t) = t$, $t \in [0,1]$, we have $Y_t = B_t - t \int_0^1 1 dB_u = B_t - tB_1$, $t \in [0,1]$, i.e., it is a Wiener bridge over $[0,1]$ from 0 to 0. However, in general, Y is not a bridge. Note that Y is a bridge in the sense that $\mathbb{P}(Y_1 = y_1) = 1$ with some $y_1 \in \mathbb{R}$ (i.e., Y takes some constant value at time 1 with probability one) if and only if $g(1) \in \{-1, 1\}$, and in this case $y_1 = 0$. Indeed, $\mathbb{P}(Y_1 = y_1) = 1$ with some $y_1 \in \mathbb{R}$ if and only if $\mathbb{D}^2(Y_1) = 0$. Since $\mathbb{D}^2(Y_1) = 1 - (g(1))^2$ (see Proposition 1.1), we have $\mathbb{P}(Y_1 = y_1) = 1$ with some $y_1 \in \mathbb{R}$ if and only if $g(1) \in \{-1, 1\}$, as desired. Further, since $\mathbb{E}(Y_1) = 0$, in this case we have $y_1 = 0$. In the present paper, we do not intend to study whether the process $(Y_t)_{t \in [0,1]}$ given by (1.1) can be considered as a bridge in the sense that it can be derived from some appropriate stochastic process (for more information on this procedure, see Barczy and Kern [5]).

To give a possible *formal* motivation for the definition of the process Y , let us write (1.1) in the form

$$dY_t = dB_t - \left(\int_0^1 g'(u) dB_u \right) g'(t) dt, \quad t \in [0,1],$$

where $\left(\int_0^1 g'(u) dB_u \right) g'(t)$ can be *formally* interpreted as the orthogonal projection of the derivative of B_t (in notation dB_t) onto g' in L^2 , since $\int_0^1 (g'(u))^2 du = 1$ (it is only a *formal* one because the derivative of B does not exist). So, from this point of view, the derivative of Y_t (in notation dY_t) is *formally* the orthogonal component of dB_t with respect to g' in L^2 , and one can call dY_t as the g' -detrendization of dB_t .

Further, we point out that if g additionally satisfies $g'(1) = 0$, then the Gauss process $(Y_t)_{t \in [0,1]}$ given in (1.1) coincides in law with one of the Gauss processes introduced in Nazarov [16, formula (1.3)], for more details, see Appendix A. In the spirit of Nazarov [16], one can say that $(Y_t)_{t \in [0,1]}$ is a perturbation of the Wiener process $(B_t)_{t \in [0,1]}$ by the function g .

1.1 Proposition. *The process $(Y_t)_{t \in [0,1]}$ is a zero-mean Gauss process with continuous sample paths almost surely and with covariance function $R(s, t) := \text{Cov}(Y_s, Y_t) = s \wedge t - g(s)g(t)$, $s, t \in [0, 1]$.*

The proof of Proposition 1.1 can be found in Section 2.

The continuity of the covariance function R yields that $(Y_t)_{t \in [0,1]}$ is L^2 -continuous, see, e.g., Theorem 1.3.4 in Ash and Gardner [3]. We also have $R \in L^2([0,1]^2)$. So, the integral operator associated to the kernel function R , i.e., the operator $A_R : L^2([0,1]) \rightarrow L^2([0,1])$,

$$(1.2) \quad (A_R(\phi))(t) := \int_0^1 R(t, s) \phi(s) ds, \quad t \in [0,1], \quad \phi \in L^2[0,1],$$

is of the Hilbert–Schmidt type, thus $(Y_t)_{t \in [0,1]}$ has a Karhunen–Loève (KL) expansion based on $[0, 1]$:

$$(1.3) \quad Y_t = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \xi_k e_k(t), \quad t \in [0,1],$$

where ξ_k , $k \in \mathbb{N}$, are independent standard normally distributed random variables, λ_k , $k \in \mathbb{N}$, are the non-zero (and hence positive) eigenvalues of the integral operator A_R and $e_k(t)$, $t \in [0,1]$, $k \in \mathbb{N}$, are the corresponding normed eigenfunctions, which are pairwise orthogonal in $L^2([0,1])$, see, e.g., Ash and Gardner [3, Theorem 1.4.1]. For completeness, we recall that the integral operator A_R

has at most countably many eigenvalues, all non-negative (due to positive semi-definiteness) with 0 as the only possible limit point, and the eigenspaces corresponding to positive eigenvalues are finite dimensional. Observe that (1.3) has infinitely many terms. Indeed, if it had a finite number of terms, i.e., if there were only a finite number of eigenfunctions, say N , then, by the help of (1.1), we would obtain that the Wiener process $(B_t)_{t \in [0,1]}$ is concentrated in an at most $(N+1)$ -dimensional subspace of $L^2([0,1])$, and so even of $C([0,1])$, with probability one. This results in a contradiction, since the integral operator associated to the covariance function (as a kernel function) of a standard Wiener process has infinitely many eigenvalues and eigenfunctions. We also note that the normed eigenfunctions are unique only up to sign. The series in (1.3) converges in $L^2(\Omega, \mathcal{A}, \mathbb{P})$ to Y_t , uniformly on $[0,1]$, i.e.,

$$\sup_{t \in [0,1]} \mathbb{E} \left(\left| Y_t - \sum_{k=1}^n \sqrt{\lambda_k} \xi_k e_k(t) \right|^2 \right) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Moreover, since R is continuous on $[0,1]^2$, the eigenfunctions corresponding to non-zero eigenvalues are also continuous on $[0,1]$, see, e.g. Ash and Gardner [3, p. 38] (this will be important in the proof of Proposition 1.2, too). Since the terms on the right-hand side of (1.3) are independent normally distributed random variables and $(Y_t)_{t \in [0,1]}$ has continuous sample paths with probability one, the series converges even uniformly on $[0,1]$ with probability one (see, e.g., Adler [1, Theorem 3.8]).

1.2 Proposition. *If λ is a non-zero (and hence positive) eigenvalue of the integral operator A_R and e is an eigenfunction corresponding to it, then*

$$(1.4) \quad \lambda e''(t) = -e(t) - g''(t) \int_0^1 g(s)e(s) ds, \quad t \in [0,1],$$

with boundary conditions

$$(1.5) \quad e(0) = 0 \quad \text{and} \quad \lambda e'(1) = -g'(1) \int_0^1 g(s)e(s) ds.$$

Conversely, if λ and $e(t)$, $t \in [0,1]$, satisfy (1.4) and (1.5), then λ is an eigenvalue of A_R and e is an eigenfunction corresponding to it.

Note that for the converse statement in Proposition 1.2 we do not need to know in advance that λ is non-zero. The proof of Proposition 1.2 can be found in Section 2.

To describe the solutions of (1.4) and (1.5), for a fixed $\lambda > 0$ we introduce the notations

$$(1.6) \quad \begin{aligned} a_g(\lambda) &:= \int_0^1 g(t) \cos\left(\frac{t}{\sqrt{\lambda}}\right) dt, & b_g(\lambda) &:= \int_0^1 g(t) \sin\left(\frac{t}{\sqrt{\lambda}}\right) dt, \\ c_g(\lambda) &:= \int_0^1 \left(\int_0^t g(u)g(t) \sin\left(\frac{u}{\sqrt{\lambda}}\right) \cos\left(\frac{t}{\sqrt{\lambda}}\right) du \right) dt. \end{aligned}$$

1.3 Theorem. *In the KL expansion (1.3) of the process $(Y_t)_{t \in [0,1]}$ given in (1.1), the non-zero (and hence positive) eigenvalues are the solutions of the equation*

$$(1.7) \quad \left(\lambda^{3/2} + \sqrt{\lambda} \int_0^1 g(t)^2 dt + 2c_g(\lambda) \right) \cos\left(\frac{1}{\sqrt{\lambda}}\right) + b_g(\lambda)^2 \sin\left(\frac{1}{\sqrt{\lambda}}\right) = 0,$$

and the corresponding normed eigenfunctions take the form

$$\begin{aligned}
(1.8) \quad e(t) = C & \left[\sqrt{\lambda} \cos\left(\frac{1}{\sqrt{\lambda}}\right) g(t) + \left(a_g(\lambda) \cos\left(\frac{1}{\sqrt{\lambda}}\right) + b_g(\lambda) \sin\left(\frac{1}{\sqrt{\lambda}}\right) \right) \sin\left(\frac{t}{\sqrt{\lambda}}\right) \right. \\
& + \cos\left(\frac{1}{\sqrt{\lambda}}\right) \cos\left(\frac{t}{\sqrt{\lambda}}\right) \int_0^t g(u) \sin\left(\frac{u}{\sqrt{\lambda}}\right) du \\
& \left. - \cos\left(\frac{1}{\sqrt{\lambda}}\right) \sin\left(\frac{t}{\sqrt{\lambda}}\right) \int_0^t g(u) \cos\left(\frac{u}{\sqrt{\lambda}}\right) du \right], \quad t \in [0, 1],
\end{aligned}$$

where $C \in \mathbb{R}$ is chosen such that $\int_0^1 (e(t))^2 dt = 1$. (Note that C may depend on λ , but we do not denote this dependence.)

The proof of Theorem 1.3 can be found in Section 2. We emphasize that in Theorem 1.3 we give KL expansion (1.3) for a new class of Gauss processes with the advantage of an explicit form of the eigenfunctions appearing in (1.3), while in the recent papers on KL expansions such as for detrended Wiener processes (Ai, Li and Liu [2]), additive Wiener processes and bridges (Liu [13]) and additive Slepian processes (Liu, Huang and Mao [14]), the form of the eigenfunctions remains somewhat hidden. As we have already mentioned, in the case of $g'(1) = 0$, the Gauss process $(Y_t)_{t \in [0,1]}$ coincides in law with one of the Gauss processes (1.3) in Nazarov [16], where he presented a procedure for finding the KL expansion for his more general Gauss processes. In Theorem 1.3 we make the KL expansion of $(Y_t)_{t \in [0,1]}$ as explicit as possible by solving the underlying eigenvalue problem directly. We note that Theorem 1.3 is applicable in the case of $g'(1) \neq 0$ as well.

1.4 Remark. Note that 0 may be an eigenvalue of the integral operator A_R defined in (1.2), which is in accordance with Corollary 2 in Nazarov [16]. For an example, see Section 2. \square

In the next remark we recall an application of the KL expansion (1.3).

1.5 Remark. The Laplace transform of the $L^2([0, 1])$ -norm square of $(Y_t)_{t \in [0,1]}$ takes the form

$$(1.9) \quad \mathbb{E} \left(\exp \left\{ -c \int_0^1 Y_t^2 dt \right\} \right) = \prod_{k=1}^{\infty} \frac{1}{\sqrt{1 + 2c\lambda_k}}, \quad c \geq 0.$$

Indeed, by (1.3), we have

$$Y_t^2 = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \sqrt{\lambda_k \lambda_{\ell}} \xi_k \xi_{\ell} e_k(t) e_{\ell}(t), \quad t \in [0, 1],$$

and hence using the fact that $(e_k)_{k \in \mathbb{N}}$ is an orthonormal system in $L^2([0, 1])$, we get

$$\int_0^1 Y_t^2 dt = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \sqrt{\lambda_k \lambda_{\ell}} \xi_k \xi_{\ell} \int_0^1 e_k(t) e_{\ell}(t) dt = \sum_{k=1}^{\infty} \lambda_k \xi_k^2,$$

which is nothing else but the Parseval identity in $L^2([0, 1])$. Since ξ_k , $k \in \mathbb{N}$, are independent standard normally distributed random variables, we get

$$\mathbb{E} \left(\exp \left\{ -c \int_0^1 Y_t^2 dt \right\} \right) = \prod_{k=1}^{\infty} \mathbb{E} \left(e^{-c\lambda_k \xi_k^2} \right) = \prod_{k=1}^{\infty} \frac{1}{\sqrt{1 + 2c\lambda_k}}, \quad c \geq 0.$$

\square

Next we study the special case $g(t) := \frac{\sqrt{2}}{\pi} \sin(\pi t)$, $t \in [0, 1]$, yielding $\mathbb{P}(Y_0 = 0) = \mathbb{P}(Y_1 = 0) = 1$.

1.6 Corollary. *If $g(t) := \frac{\sqrt{2}}{\pi} \sin(\pi t)$, $t \in [0, 1]$, then in the KL expansion (1.3) of $Y_t = B_t - \frac{2}{\pi} \sin(\pi t) \int_0^1 \cos(\pi u) dB_u$, $t \in [0, 1]$, the non-zero (and hence positive) eigenvalues are the solutions of the equation*

$$(1.10) \quad \lambda^{3/2} \cos\left(\frac{1}{\sqrt{\lambda}}\right) + \frac{2}{\pi^2 \left(\pi^2 - \frac{1}{\lambda}\right)} \sin\left(\frac{1}{\sqrt{\lambda}}\right) = 0,$$

and the corresponding normed eigenfunctions take the form

$$(1.11) \quad e(t) = C \left[\sin\left(\frac{t}{\sqrt{\lambda}}\right) - \frac{2 \sin\left(\frac{1}{\sqrt{\lambda}}\right)}{\lambda \pi \left(\pi^2 - \frac{1}{\lambda}\right)} \sin(\pi t) \right] = C \left[\sin\left(\frac{t}{\sqrt{\lambda}}\right) + \sqrt{\lambda} \pi \cos\left(\frac{1}{\sqrt{\lambda}}\right) \sin(\pi t) \right]$$

for $t \in [0, 1]$, where $C \in \mathbb{R}$ is chosen such that $\int_0^1 (e(t))^2 dt = 1$, i.e.,

$$C = \pm \left(\frac{\lambda \pi^2}{2} \cos^2\left(\frac{1}{\sqrt{\lambda}}\right) + \sqrt{\lambda} \left(\frac{\pi^2}{\pi^2 - \frac{1}{\lambda}} - \frac{1}{4} \right) \sin\left(\frac{2}{\sqrt{\lambda}}\right) + \frac{1}{2} \right)^{-\frac{1}{2}}.$$

The proof of Corollary 1.6 can be found in Section 2. In fact, we will provide two proofs. The first one is an application of Theorem 1.3, which is based on the method of variation of parameters, while the second proof is based on the method of undetermined coefficients.

1.7 Remark. The equation (1.10) has a root in every interval $\left(\frac{1}{(k+1)^2 \pi^2}, \frac{1}{k^2 \pi^2}\right)$, $k \in \mathbb{N}$. Indeed, the left hand side of (1.10) at $\lambda = \frac{1}{(k+1)^2 \pi^2}$ is equal to $(-1)^{k+1} \frac{1}{(k+1)^3 \pi^3}$, and the right hand side of (1.10) at $\lambda = \frac{1}{k^2 \pi^2}$ is equal to $(-1)^k \frac{1}{k^3 \pi^3}$, and consequently, the continuous function of the left hand side of (1.10) as a function of $\lambda > 0$ changes sign and hence has a root on every interval $\left(\frac{1}{(k+1)^2 \pi^2}, \frac{1}{k^2 \pi^2}\right)$, $k \in \mathbb{N}$. The equation (1.10) has no root greater than $\frac{4}{\pi^2}$, since

$$\lambda^{3/2} \cos\left(\frac{1}{\sqrt{\lambda}}\right) + \frac{2}{\pi^2 \left(\pi^2 - \frac{1}{\lambda}\right)} \sin\left(\frac{1}{\sqrt{\lambda}}\right) > \frac{2^3}{\pi^3} \cos\left(\frac{\pi}{2}\right) = 0, \quad \lambda \in \left(\frac{4}{\pi^2}, \infty\right).$$

Since $\frac{1}{k^2 \pi^2}$, $k \in \mathbb{N}$, are the eigenvalues of the integral operator corresponding to the covariance function $s \wedge t$, $s, t \in [0, 1]$, of a standard Wiener process, we can say that there is a kind of interlacement between the eigenvalues of the integral operators corresponding to the underlying standard Wiener process B and to the perturbed process Y . For more details on this phenomenon in a general setup, see, e.g., Nazarov [16, page 205]. Using the rootSolve package in R, we determined the first five roots of (1.10) listed in decreasing order:

$$0.338650021, \quad 0.101330775, \quad 0.021632817, \quad 0.010325434, \quad 0.006001452.$$

In Figure 1, we plotted the left hand side of (1.10) as a function of $\lambda \in (0, 0.35)$. Using the above five roots, by (1.9), for small values of $c \in (0, 1)$, the Laplace transform $\mathbb{E}\left(\exp\left\{-c \int_0^1 Y_t^2 dt\right\}\right)$ of $\int_0^1 Y_t^2 dt$, can be approximated by

$$\left(1 + 0.95588099c + 0.20572949c^2 + 0.0118929c^3 + 0.00023738c^4 + 0.00000147c^5\right)^{-\frac{1}{2}}.$$

□

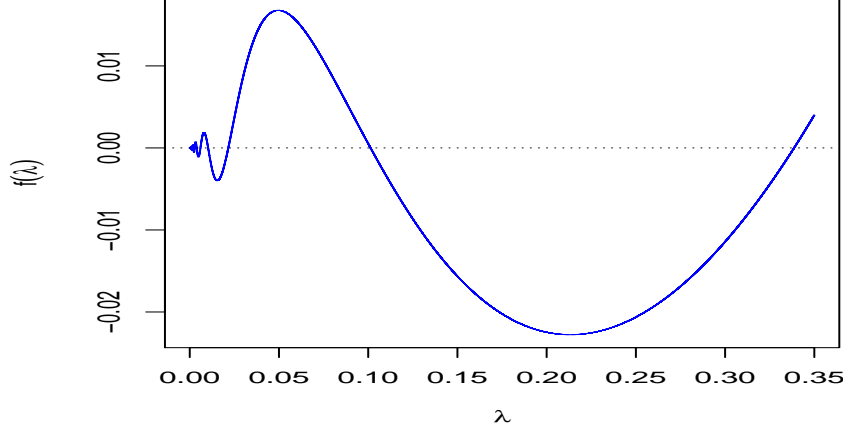


Figure 1: The function $\lambda^{3/2} \cos\left(\frac{1}{\sqrt{\lambda}}\right) + \frac{2}{\pi^2(\pi^2 - \frac{1}{\lambda})} \sin\left(\frac{1}{\sqrt{\lambda}}\right)$, $\lambda > 0$.

Finally, we study the special case $g(t) := t$, $t \in [0, 1]$, which is nothing else but the case of a usual Wiener bridge over $[0, 1]$ from 0 to 0. Note that the KL expansion of a Wiener bridge has been known for a long time, see, e.g., Deheuvels [8, Remark 1.1].

1.8 Corollary. *If $g(t) := t$, $t \in [0, 1]$, then in the KL expansion (1.3) of the Wiener bridge $Y_t = B_t - tB_1$, $t \in [0, 1]$, the non-zero (and hence positive) eigenvalues are the solutions of the equation*

$$(1.12) \quad \sin\left(\frac{1}{\sqrt{\lambda}}\right) = 0, \quad i.e., \quad \lambda = \frac{1}{(k\pi)^2}, \quad k \in \mathbb{N},$$

and the corresponding normed eigenfunctions take the form

$$(1.13) \quad e(t) = \pm\sqrt{2} \sin(k\pi t), \quad t \in [0, 1],$$

satisfying $\int_0^1 (e(t))^2 dt = 1$.

The proof of Corollary 1.8 can be found in Section 2.

2 Proofs

Proof of Proposition 1.1. The fact that Y is a zero-mean Gauss process with continuous sample paths almost surely follows from its definition. Indeed, since for all $0 \leq t_1 < t_2 < \dots < t_n$, $n \in \mathbb{N}$,

$$\begin{bmatrix} Y_{t_1} \\ \vdots \\ Y_{t_n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 & -g(t_1) \\ 0 & 1 & \cdots & 0 & -g(t_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -g(t_n) \end{bmatrix} \begin{bmatrix} B_{t_1} \\ B_{t_2} \\ \vdots \\ B_{t_n} \\ \int_0^1 g'(u) dB_u \end{bmatrix},$$

to check that Y is a Gauss process it is enough to show that

$$\left[B_{t_1} \quad \cdots \quad B_{t_n} \quad \int_0^1 g'(u) dB_u \right]$$

is normally distributed for all $0 \leq t_1 < t_2 < \cdots < t_n$, $n \in \mathbb{N}$. This follows from the fact that B is a Gauss process and from the definition of $\int_0^1 g'(u) dB_u$ taking into account that an L^2 -limit of normally distributed random variables is normally distributed (for a more detailed discussion on a similar procedure, see, e.g., the proof of Lemma 48.2 in Bauer [6]). Further, since $\int_0^t (g'(u))^2 du \leq 1$, $t \in [0, 1]$, the process $\left(\int_0^t g'(u) dB_u \right)_{t \in [0, 1]}$ is a martingale, and consequently $\mathbb{E} \left(\int_0^1 g'(u) dB_u \right) = 0$, yielding that $\mathbb{E}(Y_t) = 0$, $t \in [0, 1]$. Moreover, for $s, t \in [0, 1]$,

$$\begin{aligned} R(s, t) &= \text{Cov} \left(B_s - g(s) \int_0^1 g'(u) dB_u, B_t - g(t) \int_0^1 g'(u) dB_u \right) \\ &= \text{Cov}(B_s, B_t) - g(t) \text{Cov} \left(\int_0^s 1 dB_u, \int_0^1 g'(u) dB_u \right) \\ &\quad - g(s) \text{Cov} \left(\int_0^1 g'(u) dB_u, \int_0^t 1 dB_u \right) \\ &\quad + g(s)g(t) \text{Cov} \left(\int_0^1 g'(u) dB_u, \int_0^1 g'(u) dB_u \right) \\ &= s \wedge t - g(t) \int_0^s g'(u) du - g(s) \int_0^t g'(u) du + g(s)g(t) \int_0^1 (g'(u))^2 du \\ &= s \wedge t - g(t)g(s) - g(s)g(t) + g(s)g(t) = s \wedge t - g(s)g(t), \end{aligned}$$

where for the last but one equality we used $g(0) = 0$ and $\int_0^1 (g'(u))^2 du = 1$. \square

Proof of Proposition 1.2. Let λ be a non-zero (and hence positive) eigenvalue of the integral operator A_R . Then we have

$$(2.1) \quad \int_0^1 R(t, s)e(s) ds = \lambda e(t), \quad t \in [0, 1],$$

and hence

$$\int_0^t R(t, s)e(s) ds + \int_t^1 R(t, s)e(s) ds = \lambda e(t), \quad t \in [0, 1].$$

Then

$$\begin{aligned} (2.2) \quad \lambda e(t) &= \int_0^t (s - g(s)g(t))e(s) ds + \int_t^1 (t - g(s)g(t))e(s) ds \\ &= \int_0^t se(s) ds + t \int_t^1 e(s) ds - g(t) \int_0^1 g(s)e(s) ds, \quad t \in [0, 1]. \end{aligned}$$

The right-hand (and hence the left-hand) side of (2.2) is differentiable with respect to t , since e is continuous (see the Introduction), and, by differentiating with respect to t , we have

$$\lambda e'(t) = te(t) + \int_t^1 e(s) ds - te(t) - g'(t) \int_0^1 g(s)e(s) ds, \quad t \in [0, 1],$$

yielding that

$$(2.3) \quad \lambda e'(t) = -g'(t) \int_0^1 g(s)e(s) ds + \int_t^1 e(s) ds, \quad t \in [0, 1].$$

Differentiating (2.3) with respect to t yields (1.4) (the differentiation is allowed, since g is twice continuously differentiable). With the special choice $t = 0$ in (2.2), using that $g(0) = 0$ and $\lambda > 0$, we have the boundary condition $e(0) = 0$, yielding the first part of (1.5). Further, by (2.3) with $t = 1$, we have

$$\lambda e'(1) = -g'(1) \int_0^1 g(s)e(s) ds,$$

yielding the second part of (1.5).

Conversely, let us suppose that λ and $e(t)$, $t \in [0, 1]$, satisfy (1.4) and (1.5). Then integration of (1.4) from t to 1 gives

$$\lambda \int_t^1 e''(s) ds = - \int_t^1 e(s) ds - \int_t^1 g''(s) ds \int_0^1 g(s)e(s) ds, \quad t \in [0, 1],$$

i.e.,

$$\lambda(e'(1) - e'(t)) = - \int_t^1 e(s) ds - (g'(1) - g'(t)) \int_0^1 g(s)e(s) ds, \quad t \in [0, 1].$$

By (1.5), we have

$$-g'(1) \int_0^1 g(s)e(s) ds - \lambda e'(t) = - \int_t^1 e(s) ds - (g'(1) - g'(t)) \int_0^1 g(s)e(s) ds, \quad t \in [0, 1],$$

i.e.,

$$-\lambda e'(t) = - \int_t^1 e(s) ds + g'(t) \int_0^1 g(s)e(s) ds, \quad t \in [0, 1],$$

which is nothing else but (2.3). Integration of (2.3) from 0 to t gives

$$\lambda \int_0^t e'(s) ds = - \int_0^t g'(s) ds \int_0^1 g(s)e(s) ds + \int_0^t \left(\int_s^1 e(u) du \right) ds, \quad t \in [0, 1],$$

i.e., by integration by parts,

$$\lambda(e(t) - e(0)) = -(g(t) - g(0)) \int_0^1 g(s)e(s) ds + \int_0^t se(s) ds + t \int_t^1 e(s) ds, \quad t \in [0, 1].$$

By (1.5) and using also $g(0) = 0$, we have

$$\begin{aligned} \lambda e(t) &= -g(t) \int_0^1 g(s)e(s) ds + \int_0^t se(s) ds + t \int_t^1 e(s) ds \\ &= -g(t) \int_0^1 g(s)e(s) ds + \int_0^1 (s \wedge t)e(s) ds \\ &= \int_0^1 ((s \wedge t) - g(s)g(t))e(s) ds = \int_0^1 R(t, s)e(s) ds, \quad t \in [0, 1], \end{aligned}$$

i.e., (2.1) holds, as desired. \square

Proof of Theorem 1.3. Let $\lambda > 0$ and e be solutions of (1.4) and (1.5), and introduce the notation

$$(2.4) \quad K := \int_0^1 g(s)e(s)ds.$$

Then (1.4) and (1.5) take the form

$$(2.5) \quad \lambda e''(t) = -e(t) - Kg''(t), \quad t \in [0, 1],$$

$$(2.6) \quad e(0) = 0 \quad \text{and} \quad \lambda e'(1) = -Kg'(1),$$

respectively. These are, strictly speaking, not equations for the unknown function e and scalar λ , since e is hidden also in the coefficient K . However, it will prove convenient to consider (2.5) temporarily as a second-order linear differential equation (DE) for e . The general solution of the homogeneous part of (2.5) is

$$e(t) = c_1 \cos\left(\frac{t}{\sqrt{\lambda}}\right) + c_2 \sin\left(\frac{t}{\sqrt{\lambda}}\right), \quad t \in [0, 1],$$

where $c_1, c_2 \in \mathbb{R}$. To find the solution of the inhomogeneous equation (2.5), we use the method of variation of parameters, i.e., we are looking for e in the form

$$(2.7) \quad e(t) = c_1(t) \cos\left(\frac{t}{\sqrt{\lambda}}\right) + c_2(t) \sin\left(\frac{t}{\sqrt{\lambda}}\right), \quad t \in [0, 1],$$

with some twice continuously differentiable functions $c_1, c_2 : [0, 1] \rightarrow \mathbb{R}$. From this, we obtain the system of equations

$$\begin{aligned} \cos\left(\frac{t}{\sqrt{\lambda}}\right) c_1'(t) + \sin\left(\frac{t}{\sqrt{\lambda}}\right) c_2'(t) &= 0 \\ -\sin\left(\frac{t}{\sqrt{\lambda}}\right) c_1'(t) + \cos\left(\frac{t}{\sqrt{\lambda}}\right) c_2'(t) &= -\frac{K}{\sqrt{\lambda}} g''(t) \end{aligned}$$

for $c_1'(t)$ and $c_2'(t)$. Solving this and substituting the solutions into (2.7), we obtain

$$\begin{aligned} e(t) &= c_1 \cos\left(\frac{t}{\sqrt{\lambda}}\right) + c_2 \sin\left(\frac{t}{\sqrt{\lambda}}\right) \\ &\quad + \frac{K}{\sqrt{\lambda}} \cos\left(\frac{t}{\sqrt{\lambda}}\right) \int_0^t g''(u) \sin\left(\frac{u}{\sqrt{\lambda}}\right) du - \frac{K}{\sqrt{\lambda}} \sin\left(\frac{t}{\sqrt{\lambda}}\right) \int_0^t g''(u) \cos\left(\frac{u}{\sqrt{\lambda}}\right) du, \end{aligned}$$

where $c_1, c_2 \in \mathbb{R}$. If we take into account the initial condition $e(0) = 0$, we can write this in the form

$$e(t) = c_2 \sin\left(\frac{t}{\sqrt{\lambda}}\right) + \frac{K}{\sqrt{\lambda}} \cos\left(\frac{t}{\sqrt{\lambda}}\right) \int_0^t g''(u) \sin\left(\frac{u}{\sqrt{\lambda}}\right) du - \frac{K}{\sqrt{\lambda}} \sin\left(\frac{t}{\sqrt{\lambda}}\right) \int_0^t g''(u) \cos\left(\frac{u}{\sqrt{\lambda}}\right) du,$$

where $c_2 \in \mathbb{R}$. Applying integration by parts twice in both integrals and taking into account the condition $g(0) = 0$, from this we obtain

$$\begin{aligned} e(t) &= c_2 \sin\left(\frac{t}{\sqrt{\lambda}}\right) + \frac{K}{\sqrt{\lambda}} g'(0) \sin\left(\frac{t}{\sqrt{\lambda}}\right) - \frac{K}{\lambda} g(t) \\ &\quad + \frac{K}{\lambda^{3/2}} \sin\left(\frac{t}{\sqrt{\lambda}}\right) \int_0^t g(u) \cos\left(\frac{u}{\sqrt{\lambda}}\right) du - \frac{K}{\lambda^{3/2}} \cos\left(\frac{t}{\sqrt{\lambda}}\right) \int_0^t g(u) \sin\left(\frac{u}{\sqrt{\lambda}}\right) du. \end{aligned}$$

With the notation $A := c_2 + \frac{K}{\sqrt{\lambda}}g'(0)$ the first two terms can be contracted into one:

$$(2.8) \quad \begin{aligned} e(t) = & A \sin\left(\frac{t}{\sqrt{\lambda}}\right) - \frac{K}{\lambda}g(t) + \frac{K}{\lambda^{3/2}} \sin\left(\frac{t}{\sqrt{\lambda}}\right) \int_0^t g(u) \cos\left(\frac{u}{\sqrt{\lambda}}\right) du \\ & - \frac{K}{\lambda^{3/2}} \cos\left(\frac{t}{\sqrt{\lambda}}\right) \int_0^t g(u) \sin\left(\frac{u}{\sqrt{\lambda}}\right) du. \end{aligned}$$

Now we substitute this into the definition (2.4) of K :

$$\begin{aligned} K = \int_0^1 g(t)e(t)dt = & A \int_0^1 g(t) \sin\left(\frac{t}{\sqrt{\lambda}}\right) dt - \frac{K}{\lambda} \int_0^1 g(t)^2 dt \\ & + \frac{K}{\lambda^{3/2}} \int_0^1 \int_0^t g(u)g(t) \cos\left(\frac{u}{\sqrt{\lambda}}\right) \sin\left(\frac{t}{\sqrt{\lambda}}\right) du dt \\ & - \frac{K}{\lambda^{3/2}} \int_0^1 \int_0^t g(u)g(t) \sin\left(\frac{u}{\sqrt{\lambda}}\right) \cos\left(\frac{t}{\sqrt{\lambda}}\right) du dt, \end{aligned}$$

where

$$\begin{aligned} & \int_0^1 \int_0^t g(u)g(t) \cos\left(\frac{u}{\sqrt{\lambda}}\right) \sin\left(\frac{t}{\sqrt{\lambda}}\right) du dt \\ &= \int_0^1 \int_0^1 g(u)g(t) \cos\left(\frac{u}{\sqrt{\lambda}}\right) \sin\left(\frac{t}{\sqrt{\lambda}}\right) du dt - \int_0^1 \left(\int_t^1 g(u)g(t) \cos\left(\frac{u}{\sqrt{\lambda}}\right) \sin\left(\frac{t}{\sqrt{\lambda}}\right) du \right) dt \\ &= \int_0^1 g(u) \cos\left(\frac{u}{\sqrt{\lambda}}\right) du \int_0^1 g(t) \sin\left(\frac{t}{\sqrt{\lambda}}\right) dt - \int_0^1 \left(\int_0^u g(u)g(t) \cos\left(\frac{u}{\sqrt{\lambda}}\right) \sin\left(\frac{t}{\sqrt{\lambda}}\right) dt \right) du \\ &= a_g(\lambda)b_g(\lambda) - c_g(\lambda). \end{aligned}$$

Putting all these together, we obtain the equation

$$(2.9) \quad b_g(\lambda)A + \left(-1 - \frac{1}{\lambda} \int_0^1 g(t)^2 dt + \frac{a_g(\lambda)b_g(\lambda) - 2c_g(\lambda)}{\lambda^{3/2}} \right) K = 0.$$

Now we want to substitute e into the second equation of (2.6), therefore we calculate the derivative of (2.8):

$$\begin{aligned} e'(t) = & \frac{A}{\sqrt{\lambda}} \cos\left(\frac{t}{\sqrt{\lambda}}\right) - \frac{K}{\lambda}g'(t) \\ & + \frac{K}{\lambda^2} \cos\left(\frac{t}{\sqrt{\lambda}}\right) \int_0^t g(u) \cos\left(\frac{u}{\sqrt{\lambda}}\right) du + \frac{K}{\lambda^{3/2}} \sin\left(\frac{t}{\sqrt{\lambda}}\right) g(t) \cos\left(\frac{t}{\sqrt{\lambda}}\right) \\ & + \frac{K}{\lambda^2} \sin\left(\frac{t}{\sqrt{\lambda}}\right) \int_0^t g(u) \sin\left(\frac{u}{\sqrt{\lambda}}\right) du - \frac{K}{\lambda^{3/2}} \cos\left(\frac{t}{\sqrt{\lambda}}\right) g(t) \sin\left(\frac{t}{\sqrt{\lambda}}\right) \\ = & \frac{A}{\sqrt{\lambda}} \cos\left(\frac{t}{\sqrt{\lambda}}\right) - \frac{K}{\lambda}g'(t) + \frac{K}{\lambda^2} \cos\left(\frac{t}{\sqrt{\lambda}}\right) \int_0^t g(u) \cos\left(\frac{u}{\sqrt{\lambda}}\right) du \\ & + \frac{K}{\lambda^2} \sin\left(\frac{t}{\sqrt{\lambda}}\right) \int_0^t g(u) \sin\left(\frac{u}{\sqrt{\lambda}}\right) du, \quad t \in [0, 1]. \end{aligned}$$

Substituting this into the second equation of (2.6), we get

$$A\lambda^{3/2}\cos\left(\frac{1}{\sqrt{\lambda}}\right) + K\cos\left(\frac{1}{\sqrt{\lambda}}\right)\int_0^1 g(u)\cos\left(\frac{u}{\sqrt{\lambda}}\right)du + K\sin\left(\frac{1}{\sqrt{\lambda}}\right)\int_0^1 g(u)\sin\left(\frac{u}{\sqrt{\lambda}}\right)du = 0,$$

or, using again the notations (1.6),

$$(2.10) \quad \lambda^{3/2}\cos\left(\frac{1}{\sqrt{\lambda}}\right)A + \left(a_g(\lambda)\cos\left(\frac{1}{\sqrt{\lambda}}\right) + b_g(\lambda)\sin\left(\frac{1}{\sqrt{\lambda}}\right)\right)K = 0.$$

This, together with (2.9), yields the following homogeneous system of linear equations for the unknowns A and K :

$$(2.11) \quad \begin{aligned} b_g(\lambda)A + \left(-1 - \frac{1}{\lambda}\int_0^1 g(t)^2 dt + \frac{a_g(\lambda)b_g(\lambda) - 2c_g(\lambda)}{\lambda^{3/2}}\right)K &= 0, \\ \lambda^{3/2}\cos\left(\frac{1}{\sqrt{\lambda}}\right)A + \left(a_g(\lambda)\cos\left(\frac{1}{\sqrt{\lambda}}\right) + b_g(\lambda)\sin\left(\frac{1}{\sqrt{\lambda}}\right)\right)K &= 0. \end{aligned}$$

In what follows, we show that excluding two special cases, namely, $g(t) = t$, $t \in [0, 1]$, and $g(t) = -t$, $t \in [0, 1]$, the function e given in (2.8) can be identically zero if and only if $A = K = 0$. To prove this, it is enough to check that the functions $\sin\left(\frac{t}{\sqrt{\lambda}}\right)$, $t \in [0, 1]$, and

$$(2.12) \quad \begin{aligned} &-\frac{1}{\lambda}g(t) + \frac{1}{\lambda^{3/2}}\sin\left(\frac{t}{\sqrt{\lambda}}\right)\int_0^t g(u)\cos\left(\frac{u}{\sqrt{\lambda}}\right)du \\ &-\frac{1}{\lambda^{3/2}}\cos\left(\frac{t}{\sqrt{\lambda}}\right)\int_0^t g(u)\sin\left(\frac{u}{\sqrt{\lambda}}\right)du, \quad t \in [0, 1], \end{aligned}$$

are linearly independent. On the contrary, let us suppose that they are linearly dependent, i.e., there exist constants $\tilde{A}, \tilde{K} \in \mathbb{R}$ such that $\tilde{A}^2 + \tilde{K}^2 \neq 0$ and

$$(2.13) \quad \begin{aligned} \tilde{A}\sin\left(\frac{t}{\sqrt{\lambda}}\right) + \tilde{K}\left(-\frac{1}{\lambda}g(t) + \frac{1}{\lambda^{3/2}}\sin\left(\frac{t}{\sqrt{\lambda}}\right)\int_0^t g(u)\cos\left(\frac{u}{\sqrt{\lambda}}\right)du \right. \\ \left. - \frac{1}{\lambda^{3/2}}\cos\left(\frac{t}{\sqrt{\lambda}}\right)\int_0^t g(u)\sin\left(\frac{u}{\sqrt{\lambda}}\right)du\right) = 0, \quad t \in [0, 1]. \end{aligned}$$

By differentiating twice, one can check that

$$(2.14) \quad \begin{aligned} \tilde{A}\sin\left(\frac{t}{\sqrt{\lambda}}\right) &= -\tilde{K}\left(g''(t) - \frac{1}{\lambda}g(t) + \frac{1}{\lambda^{3/2}}\sin\left(\frac{t}{\sqrt{\lambda}}\right)\int_0^t g(u)\cos\left(\frac{u}{\sqrt{\lambda}}\right)du \right. \\ &\quad \left. - \frac{1}{\lambda^{3/2}}\cos\left(\frac{t}{\sqrt{\lambda}}\right)\int_0^t g(u)\sin\left(\frac{u}{\sqrt{\lambda}}\right)du\right), \quad t \in [0, 1]. \end{aligned}$$

Comparing (2.13) and (2.14), we have $\tilde{K}g''(t) = 0$, $t \in [0, 1]$. If $\tilde{K} = 0$, then $\sin\left(\frac{t}{\sqrt{\lambda}}\right) = 0$, $t \in [0, 1]$, which is a contradiction. Thus $\tilde{K} \neq 0$, and we have $g''(t) = 0$, $t \in [0, 1]$. Using that $g(0) = 0$ and $\int_0^1 (g'(t))^2 dt = 1$, we get $g(t) = t$, $t \in [0, 1]$ or $g(t) = -t$, $t \in [0, 1]$, which cases were excluded. This leads us to a contradiction.

Hence, excluding the two special cases $g(t) = t, t \in [0, 1]$, and $g(t) = -t, t \in [0, 1]$, we see that e is not identically zero if and only if at least one of the two coefficients A and K is different from zero, i.e., the system (2.11) has a nontrivial solution for A and K . This is equivalent to the condition that its determinant is zero, which in turn yields equation (1.7).

If $g(t) = t, t \in [0, 1]$, or $g(t) = -t, t \in [0, 1]$, then, by integration by parts, we have

$$\begin{aligned} & -\frac{1}{\lambda}g(t) + \frac{1}{\lambda^{3/2}}\sin\left(\frac{t}{\sqrt{\lambda}}\right)\int_0^t g(u)\cos\left(\frac{u}{\sqrt{\lambda}}\right)du - \frac{1}{\lambda^{3/2}}\cos\left(\frac{t}{\sqrt{\lambda}}\right)\int_0^t g(u)\sin\left(\frac{u}{\sqrt{\lambda}}\right)du \\ & = \mp\left(\frac{1}{\lambda}\sin\left(\frac{t}{\sqrt{\lambda}}\right)\int_0^t \sin\left(\frac{u}{\sqrt{\lambda}}\right)du + \frac{1}{\lambda}\cos\left(\frac{t}{\sqrt{\lambda}}\right)\int_0^t \cos\left(\frac{u}{\sqrt{\lambda}}\right)du\right) \\ & = \mp\frac{1}{\sqrt{\lambda}}\sin\left(\frac{t}{\sqrt{\lambda}}\right), \quad t \in [0, 1], \end{aligned}$$

yielding that the function $\sin\left(\frac{t}{\sqrt{\lambda}}\right), t \in [0, 1]$, and the function in (2.12) are linearly dependent. Further, by (2.8), using $A = c_2 + \frac{K}{\sqrt{\lambda}}g'(0) = c_2 \pm \frac{K}{\sqrt{\lambda}}$, we have

$$\begin{aligned} e(t) &= \left(A \mp \frac{K}{\sqrt{\lambda}}\right)\sin\left(\frac{t}{\sqrt{\lambda}}\right) = c_2\sin\left(\frac{t}{\sqrt{\lambda}}\right), \quad t \in [0, 1], \\ K &= \int_0^1 g(t)e(t)dt = \pm c_2 \int_0^1 t\sin\left(\frac{t}{\sqrt{\lambda}}\right)dt = \pm c_2\sqrt{\lambda}\left(-\cos\left(\frac{1}{\sqrt{\lambda}}\right) + \sqrt{\lambda}\sin\left(\frac{1}{\sqrt{\lambda}}\right)\right). \end{aligned}$$

By the second equation of (2.6), we have $c_2\sqrt{\lambda}\cos\left(\frac{1}{\sqrt{\lambda}}\right) = \mp K$ (which is in fact (2.10) in the special cases $g(t) = t, t \in [0, 1]$, and $g(t) = -t, t \in [0, 1]$). This together with the above form of K , yields $\sin\left(\frac{1}{\sqrt{\lambda}}\right) = 0$, i.e., $\lambda = \frac{1}{(k\pi)^2}, k \in \mathbb{N}$. Next we check that the equation $\sin\left(\frac{1}{\sqrt{\lambda}}\right) = 0$ is nothing else but the equation (1.7) in the special cases $g(t) = t, t \in [0, 1]$, and $g(t) = -t, t \in [0, 1]$. Using integration by parts, the constants defined in (1.6) take the forms

$$\begin{aligned} a_g(\lambda) &= \pm \int_0^1 u\cos\left(\frac{u}{\sqrt{\lambda}}\right)du = \pm\left(\sqrt{\lambda}\sin\left(\frac{1}{\sqrt{\lambda}}\right) + \lambda\left(\cos\left(\frac{1}{\sqrt{\lambda}}\right) - 1\right)\right), \quad \lambda > 0, \\ b_g(\lambda) &= \pm \int_0^1 u\sin\left(\frac{u}{\sqrt{\lambda}}\right)du = \pm\left(-\sqrt{\lambda}\cos\left(\frac{1}{\sqrt{\lambda}}\right) + \lambda\sin\left(\frac{1}{\sqrt{\lambda}}\right)\right), \quad \lambda > 0, \\ c_g(\lambda) &= \int_0^1 \int_0^t ut\sin\left(\frac{u}{\sqrt{\lambda}}\right)\cos\left(\frac{t}{\sqrt{\lambda}}\right)du dt \\ &= -\sqrt{\lambda}\int_0^1 t^2\cos^2\left(\frac{t}{\sqrt{\lambda}}\right)dt + \lambda\int_0^1 t\sin\left(\frac{t}{\sqrt{\lambda}}\right)\cos\left(\frac{t}{\sqrt{\lambda}}\right)dt \\ &= -\frac{\sqrt{\lambda}}{6} - \frac{\lambda^{3/2}}{2}\cos\left(\frac{2}{\sqrt{\lambda}}\right) + \frac{\lambda(\lambda-1)}{4}\sin\left(\frac{2}{\sqrt{\lambda}}\right), \quad \lambda > 0. \end{aligned}$$

Hence, using $\int_0^1 (g(t))^2 dt = \int_0^1 (\pm t)^2 dt = \frac{1}{3}$, the equation (1.7) takes the form

$$\begin{aligned} & \left(\lambda^{3/2} + \frac{\sqrt{\lambda}}{3} + 2\left(-\frac{\sqrt{\lambda}}{6} - \frac{\lambda^{3/2}}{2}\cos\left(\frac{2}{\sqrt{\lambda}}\right) + \frac{\lambda(\lambda-1)}{4}\sin\left(\frac{2}{\sqrt{\lambda}}\right)\right)\right)\cos\left(\frac{1}{\sqrt{\lambda}}\right) \\ & + \left(-\sqrt{\lambda}\cos\left(\frac{1}{\sqrt{\lambda}}\right) + \lambda\sin\left(\frac{1}{\sqrt{\lambda}}\right)\right)^2\sin\left(\frac{1}{\sqrt{\lambda}}\right) = 0. \end{aligned}$$

By some algebraic transformations, it is equivalent to $\lambda^2 \sin\left(\frac{1}{\sqrt{\lambda}}\right) = 0$. Since $\lambda > 0$, we have $\sin\left(\frac{1}{\sqrt{\lambda}}\right) = 0$, yielding $\lambda = \frac{1}{(k\pi)^2}$, $k \in \mathbb{N}$, as desired.

All in all, for every possible g , the equation (1.7) holds. It remains to study the form of the eigenfunctions.

If $a_g(\lambda) \cos\left(\frac{1}{\sqrt{\lambda}}\right) + b_g(\lambda) \sin\left(\frac{1}{\sqrt{\lambda}}\right) \neq 0$, then from the second equation of (2.11) we have

$$K = -\frac{\lambda^{3/2} \cos\left(\frac{1}{\sqrt{\lambda}}\right) A}{a_g(\lambda) \cos\left(\frac{1}{\sqrt{\lambda}}\right) + b_g(\lambda) \sin\left(\frac{1}{\sqrt{\lambda}}\right)} = -\tilde{C} \lambda^{3/2} \cos\left(\frac{1}{\sqrt{\lambda}}\right),$$

where

$$\tilde{C} := \frac{A}{a_g(\lambda) \cos\left(\frac{1}{\sqrt{\lambda}}\right) + b_g(\lambda) \sin\left(\frac{1}{\sqrt{\lambda}}\right)}.$$

Finally, if we substitute these expressions for K and A into (2.8), then we obtain (1.8) with some appropriately chosen $C \in \mathbb{R}$.

If $a_g(\lambda) \cos\left(\frac{1}{\sqrt{\lambda}}\right) + b_g(\lambda) \sin\left(\frac{1}{\sqrt{\lambda}}\right) = 0$, then we also show that

$$(2.15) \quad \begin{aligned} e(t) = C & \left[\sqrt{\lambda} \cos\left(\frac{1}{\sqrt{\lambda}}\right) g(t) + \cos\left(\frac{1}{\sqrt{\lambda}}\right) \cos\left(\frac{t}{\sqrt{\lambda}}\right) \int_0^t g(u) \sin\left(\frac{u}{\sqrt{\lambda}}\right) du \right. \\ & \left. - \cos\left(\frac{1}{\sqrt{\lambda}}\right) \sin\left(\frac{t}{\sqrt{\lambda}}\right) \int_0^t g(u) \cos\left(\frac{u}{\sqrt{\lambda}}\right) du \right], \quad t \in [0, 1], \end{aligned}$$

is a normed eigenvector corresponding to the eigenvalue λ , where $C \in \mathbb{R}$ is chosen such that $\int_0^1 (e(t))^2 dt = 1$. Note that (2.15) is a special case of (1.8). By Proposition 1.2, it is enough to verify that $e(t)$, $t \in [0, 1]$, given in (2.15) satisfies (1.4) and (1.5). First, note that, by integration by parts, one can calculate

$$\int_0^1 g(s) e(s) ds = C \left[\sqrt{\lambda} \int_0^1 (g(s))^2 ds - a_g(\lambda) b_g(\lambda) + 2c_g(\lambda) \right] \cos\left(\frac{1}{\sqrt{\lambda}}\right).$$

Further,

$$\begin{aligned} e'(t) = C & \left[\sqrt{\lambda} \cos\left(\frac{1}{\sqrt{\lambda}}\right) g'(t) - \frac{1}{\sqrt{\lambda}} \cos\left(\frac{1}{\sqrt{\lambda}}\right) \sin\left(\frac{t}{\sqrt{\lambda}}\right) \int_0^t g(u) \sin\left(\frac{u}{\sqrt{\lambda}}\right) du \right. \\ & \left. - \frac{1}{\sqrt{\lambda}} \cos\left(\frac{1}{\sqrt{\lambda}}\right) \cos\left(\frac{t}{\sqrt{\lambda}}\right) \int_0^t g(u) \cos\left(\frac{u}{\sqrt{\lambda}}\right) du \right], \quad t \in [0, 1], \end{aligned}$$

and

$$\begin{aligned} e''(t) = C & \left[\sqrt{\lambda} \cos\left(\frac{1}{\sqrt{\lambda}}\right) g''(t) - \frac{1}{\sqrt{\lambda}} \cos\left(\frac{1}{\sqrt{\lambda}}\right) g(t) - \frac{1}{\lambda} \cos\left(\frac{1}{\sqrt{\lambda}}\right) \cos\left(\frac{t}{\sqrt{\lambda}}\right) \int_0^t g(u) \sin\left(\frac{u}{\sqrt{\lambda}}\right) du \right. \\ & \left. + \frac{1}{\lambda} \cos\left(\frac{1}{\sqrt{\lambda}}\right) \sin\left(\frac{t}{\sqrt{\lambda}}\right) \int_0^t g(u) \cos\left(\frac{u}{\sqrt{\lambda}}\right) du \right], \quad t \in [0, 1], \end{aligned}$$

yielding that

$$\lambda e''(t) + e(t) = C\lambda^{3/2} \cos\left(\frac{1}{\sqrt{\lambda}}\right) g''(t), \quad t \in [0, 1].$$

Hence, taking into account that $C \neq 0$, to verify (1.4) it remains to check that

$$\lambda^{3/2} \cos\left(\frac{1}{\sqrt{\lambda}}\right) g''(t) = - \left[\sqrt{\lambda} \int_0^1 (g(s))^2 ds - a_g(\lambda)b_g(\lambda) + 2c_g(\lambda) \right] \cos\left(\frac{1}{\sqrt{\lambda}}\right) g''(t), \quad t \in [0, 1],$$

which is equivalent to

$$(2.16) \quad g''(t) \left[\lambda^{3/2} + \sqrt{\lambda} \int_0^1 (g(s))^2 ds - a_g(\lambda)b_g(\lambda) + 2c_g(\lambda) \right] \cos\left(\frac{1}{\sqrt{\lambda}}\right) = 0, \quad t \in [0, 1].$$

Taking into account (1.7) and that $a_g(\lambda) \cos\left(\frac{1}{\sqrt{\lambda}}\right) + b_g(\lambda) \sin\left(\frac{1}{\sqrt{\lambda}}\right) = 0$, we have

$$(2.17) \quad \left[\lambda^{3/2} + \sqrt{\lambda} \int_0^1 (g(s))^2 ds - a_g(\lambda)b_g(\lambda) + 2c_g(\lambda) \right] \cos\left(\frac{1}{\sqrt{\lambda}}\right) = 0,$$

yielding (2.16). The boundary conditions (1.5) hold as well. Indeed, the boundary condition $e(0) = 0$ is satisfied, since $g(0) = 0$, and the boundary condition $\lambda e'(1) = -g'(1) \int_0^1 g(s)e(s) ds$ is equivalent to

$$\begin{aligned} \cos\left(\frac{1}{\sqrt{\lambda}}\right) \left[g'(1) \left(\lambda^{3/2} + \sqrt{\lambda} \int_0^1 (g(s))^2 ds - a_g(\lambda)b_g(\lambda) + 2c_g(\lambda) \right) \right. \\ \left. - \sqrt{\lambda} \left(a_g(\lambda) \cos\left(\frac{1}{\sqrt{\lambda}}\right) + b_g(\lambda) \sin\left(\frac{1}{\sqrt{\lambda}}\right) \right) \right] = 0, \end{aligned}$$

which is satisfied due to (2.17). \square

An example for the assertion in Remark 1.4. Let $g : [0, 1] \rightarrow \mathbb{R}$, $g(t) := \frac{2\sqrt{2}}{\pi} \sin\left(\frac{\pi t}{2}\right)$, $t \in [0, 1]$. Then $g(0) = 0$, $g'(1) = 0$, $\int_0^1 (g'(t))^2 dt = 1$, and 0 is an eigenvalue of A_R with $-g''(t) = \frac{\pi}{\sqrt{2}} \sin\left(\frac{\pi t}{2}\right)$ as an eigenfunction corresponding to 0, which is in accordance with Nazarov [16, Corollary 2]. Indeed,

$$\begin{aligned} \int_0^1 R(t, s)e(s) ds &= \int_0^1 \left(s \wedge t - \frac{8}{\pi^2} \sin\left(\frac{\pi s}{2}\right) \sin\left(\frac{\pi t}{2}\right) \right) \frac{\pi}{\sqrt{2}} \sin\left(\frac{\pi s}{2}\right) ds \\ &= \frac{\pi}{\sqrt{2}} \left[\int_0^t s \sin\left(\frac{\pi s}{2}\right) ds + t \int_t^1 \sin\left(\frac{\pi s}{2}\right) ds - \frac{8}{\pi^2} \sin\left(\frac{\pi t}{2}\right) \int_0^1 \sin^2\left(\frac{\pi s}{2}\right) ds \right] = 0. \end{aligned}$$

\square

First proof of Corollary 1.6. We will apply Theorem 1.3 with the function $g : [0, 1] \rightarrow \mathbb{R}$, $g(t) := \frac{\sqrt{2}}{\pi} \sin(\pi t)$, $t \in [0, 1]$. First, we check that $\lambda = \frac{1}{\pi^2}$ cannot be an eigenvalue. On the contrary, let us suppose that $\frac{1}{\pi^2}$ is an eigenvalue. Then, using $\int_0^1 (g(t))^2 dt = \frac{2}{\pi^2} \int_0^1 \sin^2(\pi t) dt = \frac{1}{\pi^2}$, (1.7) would imply that

$$\left(\frac{1}{\pi^3} + \frac{1}{\pi^3} + \frac{4}{\pi^2} \int_0^1 \left(\int_0^t \sin^2(\pi u) \sin(\pi t) \cos(\pi t) du \right) dt \right) (-1) = 0,$$

which is equivalent to $1 + \pi \int_0^1 \left(\int_0^t \sin^2(\pi u) \sin(2\pi t) du \right) dt = 0$. Calculating the integral, this leads us to $\frac{5}{8} = 0$, being a contradiction.

Using $\lambda \neq \frac{1}{\pi^2}$ and the addition formulas for cosine and sine, the constants defined in (1.6) take the forms

$$\begin{aligned} a_g(\lambda) &= \frac{\sqrt{2}}{\pi} \int_0^1 \sin(\pi u) \cos\left(\frac{u}{\sqrt{\lambda}}\right) du = \frac{1}{\sqrt{2}\pi} \int_0^1 \left[\sin\left(\left(\pi + \frac{1}{\sqrt{\lambda}}\right)u\right) + \sin\left(\left(\pi - \frac{1}{\sqrt{\lambda}}\right)u\right) \right] du \\ &= \frac{\sqrt{2}}{\pi^2 - \frac{1}{\lambda}} \left(1 + \cos\left(\frac{1}{\sqrt{\lambda}}\right) \right), \quad \lambda > 0, \\ b_g(\lambda) &= \frac{\sqrt{2}}{\pi} \int_0^1 \sin(\pi u) \sin\left(\frac{u}{\sqrt{\lambda}}\right) du = \frac{1}{\sqrt{2}\pi} \int_0^1 \left[\cos\left(\left(\pi - \frac{1}{\sqrt{\lambda}}\right)u\right) - \cos\left(\left(\pi + \frac{1}{\sqrt{\lambda}}\right)u\right) \right] du \\ &= \frac{\sqrt{2}}{\pi^2 - \frac{1}{\lambda}} \sin\left(\frac{1}{\sqrt{\lambda}}\right), \quad \lambda > 0, \end{aligned}$$

and

$$\begin{aligned} c_g(\lambda) &= \frac{2}{\pi^2} \int_0^1 \int_0^t \sin(\pi u) \sin(\pi t) \sin\left(\frac{u}{\sqrt{\lambda}}\right) \cos\left(\frac{t}{\sqrt{\lambda}}\right) du dt \\ &= \frac{1}{\pi^2} \int_0^1 \sin(\pi t) \cos\left(\frac{t}{\sqrt{\lambda}}\right) \left(\frac{\sin\left(\left(\pi - \frac{1}{\sqrt{\lambda}}\right)t\right)}{\pi - \frac{1}{\sqrt{\lambda}}} - \frac{\sin\left(\left(\pi + \frac{1}{\sqrt{\lambda}}\right)t\right)}{\pi + \frac{1}{\sqrt{\lambda}}} \right) dt \\ &= \frac{1}{2\pi^2 \left(\pi - \frac{1}{\sqrt{\lambda}}\right)} \int_0^1 \left(\sin\left(\left(\pi + \frac{1}{\sqrt{\lambda}}\right)t\right) + \sin\left(\left(\pi - \frac{1}{\sqrt{\lambda}}\right)t\right) \right) \sin\left(\left(\pi - \frac{1}{\sqrt{\lambda}}\right)t\right) dt \\ &\quad - \frac{1}{2\pi^2 \left(\pi + \frac{1}{\sqrt{\lambda}}\right)} \int_0^1 \left(\sin\left(\left(\pi + \frac{1}{\sqrt{\lambda}}\right)t\right) + \sin\left(\left(\pi - \frac{1}{\sqrt{\lambda}}\right)t\right) \right) \sin\left(\left(\pi + \frac{1}{\sqrt{\lambda}}\right)t\right) dt \\ &= \frac{1}{2\pi^2} \left[\frac{1}{\pi - \frac{1}{\sqrt{\lambda}}} \int_0^1 \sin^2\left(\left(\pi - \frac{1}{\sqrt{\lambda}}\right)t\right) dt - \frac{1}{\pi + \frac{1}{\sqrt{\lambda}}} \int_0^1 \sin^2\left(\left(\pi + \frac{1}{\sqrt{\lambda}}\right)t\right) dt \right. \\ &\quad \left. + \frac{\frac{2}{\sqrt{\lambda}}}{\pi^2 - \frac{1}{\lambda}} \int_0^1 \sin\left(\left(\pi - \frac{1}{\sqrt{\lambda}}\right)t\right) \sin\left(\left(\pi + \frac{1}{\sqrt{\lambda}}\right)t\right) dt \right] \\ &= \frac{1}{2\pi^2 \left(\pi^2 - \frac{1}{\lambda}\right)} \left[\frac{1}{\sqrt{\lambda}} + \frac{\pi^2}{\pi^2 - \frac{1}{\lambda}} \sin\left(\frac{2}{\sqrt{\lambda}}\right) \right], \quad \lambda > 0. \end{aligned}$$

Hence, using $\int_0^1 (g(t))^2 dt = \frac{1}{\pi^2}$, the equation (1.7) takes the form

$$\left(\lambda^{3/2} + \frac{\sqrt{\lambda}}{\pi^2} + \frac{1}{\pi^2 \left(\pi^2 - \frac{1}{\lambda}\right)} \left(\frac{1}{\sqrt{\lambda}} + \frac{\pi^2}{\pi^2 - \frac{1}{\lambda}} \sin\left(\frac{2}{\sqrt{\lambda}}\right) \right) \right) \cos\left(\frac{1}{\sqrt{\lambda}}\right) + \frac{2}{\left(\pi^2 - \frac{1}{\lambda}\right)^2} \sin^3\left(\frac{1}{\sqrt{\lambda}}\right) = 0,$$

which, by some algebraic transformations, yields (1.10).

Further, the normed eigenfunctions (1.8) take the form

$$\begin{aligned}
e(t) &= C \left[\frac{\sqrt{2}}{\pi} \sqrt{\lambda} \cos\left(\frac{1}{\sqrt{\lambda}}\right) \sin(\pi t) + \frac{\sqrt{2}}{\pi^2 - \frac{1}{\lambda}} \left(\left(1 + \cos\left(\frac{1}{\sqrt{\lambda}}\right)\right) \cos\left(\frac{1}{\sqrt{\lambda}}\right) + \sin^2\left(\frac{1}{\sqrt{\lambda}}\right) \right) \sin\left(\frac{t}{\sqrt{\lambda}}\right) \right. \\
&\quad + \frac{\sqrt{2}}{\pi} \cos\left(\frac{1}{\sqrt{\lambda}}\right) \cos\left(\frac{t}{\sqrt{\lambda}}\right) \int_0^t \sin(\pi u) \sin\left(\frac{u}{\sqrt{\lambda}}\right) du \\
&\quad \left. - \frac{\sqrt{2}}{\pi} \cos\left(\frac{1}{\sqrt{\lambda}}\right) \sin\left(\frac{t}{\sqrt{\lambda}}\right) \int_0^t \sin(\pi u) \cos\left(\frac{u}{\sqrt{\lambda}}\right) du \right] \\
&= C \left[\frac{\sqrt{2}}{\pi} \sqrt{\lambda} \cos\left(\frac{1}{\sqrt{\lambda}}\right) \sin(\pi t) + \frac{\sqrt{2}}{\pi^2 - \frac{1}{\lambda}} \left(1 + \cos\left(\frac{1}{\sqrt{\lambda}}\right)\right) \sin\left(\frac{t}{\sqrt{\lambda}}\right) \right. \\
&\quad + \frac{1}{\sqrt{2}\pi} \cos\left(\frac{1}{\sqrt{\lambda}}\right) \cos\left(\frac{t}{\sqrt{\lambda}}\right) \left(\frac{\sin\left(\left(\pi - \frac{1}{\sqrt{\lambda}}\right)t\right)}{\pi - \frac{1}{\sqrt{\lambda}}} - \frac{\sin\left(\left(\pi + \frac{1}{\sqrt{\lambda}}\right)t\right)}{\pi + \frac{1}{\sqrt{\lambda}}} \right) \\
&\quad \left. - \frac{1}{\sqrt{2}\pi} \cos\left(\frac{1}{\sqrt{\lambda}}\right) \sin\left(\frac{t}{\sqrt{\lambda}}\right) \left(-\frac{\cos\left(\left(\pi + \frac{1}{\sqrt{\lambda}}\right)t\right)}{\pi + \frac{1}{\sqrt{\lambda}}} - \frac{\cos\left(\left(\pi - \frac{1}{\sqrt{\lambda}}\right)t\right)}{\pi - \frac{1}{\sqrt{\lambda}}} + \frac{1}{\pi + \frac{1}{\sqrt{\lambda}}} + \frac{1}{\pi - \frac{1}{\sqrt{\lambda}}} \right) \right] \\
&= C \left[\frac{\sqrt{2}}{\pi} \sqrt{\lambda} \cos\left(\frac{1}{\sqrt{\lambda}}\right) \sin(\pi t) + \frac{\sqrt{2}}{\pi^2 - \frac{1}{\lambda}} \sin\left(\frac{t}{\sqrt{\lambda}}\right) \right. \\
&\quad + \frac{1}{\sqrt{2}\pi \left(\pi - \frac{1}{\sqrt{\lambda}}\right)} \cos\left(\frac{1}{\sqrt{\lambda}}\right) \left(\sin\left(\left(\pi - \frac{1}{\sqrt{\lambda}}\right)t\right) \cos\left(\frac{t}{\sqrt{\lambda}}\right) + \cos\left(\left(\pi - \frac{1}{\sqrt{\lambda}}\right)t\right) \sin\left(\frac{t}{\sqrt{\lambda}}\right) \right) \\
&\quad \left. + \frac{1}{\sqrt{2}\pi \left(\pi + \frac{1}{\sqrt{\lambda}}\right)} \cos\left(\frac{1}{\sqrt{\lambda}}\right) \left(\sin\left(\frac{t}{\sqrt{\lambda}}\right) \cos\left(\left(\pi + \frac{1}{\sqrt{\lambda}}\right)t\right) - \cos\left(\frac{t}{\sqrt{\lambda}}\right) \sin\left(\left(\pi + \frac{1}{\sqrt{\lambda}}\right)t\right) \right) \right] \\
&= C \left[\frac{\sqrt{2}}{\pi} \sqrt{\lambda} \cos\left(\frac{1}{\sqrt{\lambda}}\right) \sin(\pi t) + \frac{\sqrt{2}}{\pi^2 - \frac{1}{\lambda}} \sin\left(\frac{t}{\sqrt{\lambda}}\right) \right. \\
&\quad \left. + \frac{1}{\sqrt{2}\pi \left(\pi - \frac{1}{\sqrt{\lambda}}\right)} \cos\left(\frac{1}{\sqrt{\lambda}}\right) \sin(\pi t) + \frac{1}{\sqrt{2}\pi \left(\pi + \frac{1}{\sqrt{\lambda}}\right)} \cos\left(\frac{1}{\sqrt{\lambda}}\right) \sin(-\pi t) \right] \\
&= \frac{C\sqrt{2}}{\pi^2 - \frac{1}{\lambda}} \left(\sin\left(\frac{t}{\sqrt{\lambda}}\right) + \sqrt{\lambda}\pi \cos\left(\frac{1}{\sqrt{\lambda}}\right) \sin(\pi t) \right),
\end{aligned}$$

where $C \in \mathbb{R}$ is such that $\int_0^1 (e(t))^2 dt = 1$. Hence

$$\begin{aligned}
\frac{1}{C^2} &= \frac{2}{\left(\pi^2 - \frac{1}{\lambda}\right)^2} \int_0^1 \left(\sin^2\left(\frac{t}{\sqrt{\lambda}}\right) + \lambda\pi^2 \cos^2\left(\frac{1}{\sqrt{\lambda}}\right) \sin^2(\pi t) + 2\sqrt{\lambda}\pi \cos\left(\frac{1}{\sqrt{\lambda}}\right) \sin\left(\frac{t}{\sqrt{\lambda}}\right) \sin(\pi t) \right) dt \\
&= \frac{2}{\left(\pi^2 - \frac{1}{\lambda}\right)^2} \left(\frac{\lambda\pi^2}{2} \cos^2\left(\frac{1}{\sqrt{\lambda}}\right) + \sqrt{\lambda} \left(\frac{\pi^2}{\pi^2 - \frac{1}{\lambda}} - \frac{1}{4} \right) \sin\left(\frac{2}{\sqrt{\lambda}}\right) + \frac{1}{2} \right).
\end{aligned}$$

Merging the factor $\frac{\sqrt{2}}{\pi^2-1/\lambda}$ into C , taking into account (1.10) and that

$$\sqrt{\lambda}\pi \cos\left(\frac{1}{\sqrt{\lambda}}\right) = -\frac{2}{\lambda\pi\left(\pi^2 - \frac{1}{\lambda}\right)} \sin\left(\frac{1}{\sqrt{\lambda}}\right),$$

this yields (1.11). \square

Second proof of Corollary 1.6. In the special case $g(t) := \frac{\sqrt{2}}{\pi} \sin(\pi t)$, $t \in [0, 1]$, the DE (1.4) and the boundary conditions (1.5) take the form

$$(2.18) \quad \lambda e''(t) = -e(t) + 2 \sin(\pi t) \int_0^1 \sin(\pi s) e(s) ds, \quad t \in [0, 1],$$

and

$$e(0) = 0 \quad \text{and} \quad \lambda e'(1) = \frac{2}{\pi} \int_0^1 \sin(\pi s) e(s) ds,$$

respectively. With the special choice $t = 0$, using $e(0) = 0$ and $\lambda > 0$, we have $e''(0) = 0$. The DE (2.18) is a second order linear inhomogeneous DE of the type $\lambda e''(t) = -e(t) + B \sin(\pi t)$, $t \in [0, 1]$, where $B := 2 \int_0^1 \sin(\pi s) e(s) ds$ (for more details, see the proof of Theorem 1.3). By the method of undetermined coefficients, its general solution takes the form

$$(2.19) \quad e(t) = a \sin\left(\frac{t}{\sqrt{\lambda}}\right) + b \cos\left(\frac{t}{\sqrt{\lambda}}\right) + c \sin(\pi t) + d \cos(\pi t), \quad t \in [0, 1],$$

where $a, b, c, d \in \mathbb{R}$. Hence

$$(2.20) \quad e'(t) = \frac{a}{\sqrt{\lambda}} \cos\left(\frac{t}{\sqrt{\lambda}}\right) - \frac{b}{\sqrt{\lambda}} \sin\left(\frac{t}{\sqrt{\lambda}}\right) + c\pi \cos(\pi t) - d\pi \sin(\pi t), \quad t \in [0, 1],$$

$$(2.21) \quad e''(t) = -\frac{a}{\lambda} \sin\left(\frac{t}{\sqrt{\lambda}}\right) - \frac{b}{\lambda} \cos\left(\frac{t}{\sqrt{\lambda}}\right) - c\pi^2 \sin(\pi t) - d\pi^2 \cos(\pi t), \quad t \in [0, 1].$$

Then the DE (2.18) takes the form

$$(2.22) \quad c(1 - \lambda\pi^2) \sin(\pi t) + d(1 - \lambda\pi^2) \cos(\pi t) = 2 \sin(\pi t) \int_0^1 \sin(\pi s) e(s) ds, \quad t \in [0, 1].$$

Since $e(0) = 0$, by (2.19), we have $d = -b$. Since $e''(0) = 0$, by (2.21), we have $0 = -\frac{b}{\lambda} - d\pi^2$, and then, since $d = -b$, we get $b = d = 0$ or $\lambda = \frac{1}{\pi^2}$.

Next, we check that $\lambda = \frac{1}{\pi^2}$ cannot be an eigenvalue. On the contrary, let us suppose that $\lambda = \frac{1}{\pi^2}$ is an eigenvalue. Then, since $d = -b$, by (2.19), we have

$$e(t) = a \sin(\pi t) + c \sin(\pi t) = (a + c) \sin(\pi t), \quad t \in [0, 1],$$

where $a, c \in \mathbb{R}$. Further, by (2.22),

$$0 = \int_0^1 \sin(\pi s) e(s) ds = (a + c) \int_0^1 (\sin(\pi s))^2 ds = \frac{a + c}{2},$$

yielding $e(t) = 0$, $t \in [0, 1]$, which leads us to a contradiction.

Since $\lambda \neq \frac{1}{\pi^2}$, then we have $b = d = 0$. By (2.19) and (2.2) together with $g(1) = \frac{\sqrt{2}}{\pi} \sin(\pi) = 0$, we have $e(t) = a \sin\left(\frac{t}{\sqrt{\lambda}}\right) + c \sin(\pi t)$, $t \in [0, 1]$, and $\lambda e(1) = \int_0^1 s e(s) ds$. Hence

$$\begin{aligned} \lambda a \sin\left(\frac{1}{\sqrt{\lambda}}\right) &= a \int_0^1 s \sin\left(\frac{s}{\sqrt{\lambda}}\right) ds + c \int_0^1 s \sin(\pi s) ds \\ &= a \left(-\sqrt{\lambda} \cos\left(\frac{1}{\sqrt{\lambda}}\right) + \lambda \sin\left(\frac{1}{\sqrt{\lambda}}\right) \right) + \frac{c}{\pi}. \end{aligned}$$

Then

$$(2.23) \quad -a\sqrt{\lambda} \cos\left(\frac{1}{\sqrt{\lambda}}\right) + \frac{c}{\pi} = 0.$$

By (2.20) and $\lambda e'(1) = \frac{2}{\pi} \int_0^1 \sin(\pi s) e(s) ds$, we have

$$\begin{aligned} a\sqrt{\lambda} \cos\left(\frac{1}{\sqrt{\lambda}}\right) - c\lambda\pi &= \frac{2a}{\pi} \int_0^1 \sin(\pi s) \sin\left(\frac{s}{\sqrt{\lambda}}\right) ds + \frac{2c}{\pi} \int_0^1 (\sin(\pi s))^2 ds \\ &= \frac{a}{\pi} \left(\frac{1}{\pi - \frac{1}{\sqrt{\lambda}}} \sin\left(\pi - \frac{1}{\sqrt{\lambda}}\right) - \frac{1}{\pi + \frac{1}{\sqrt{\lambda}}} \sin\left(\pi + \frac{1}{\sqrt{\lambda}}\right) \right) + \frac{c}{\pi}, \end{aligned}$$

and hence

$$\begin{aligned} (2.24) \quad &\left(\sqrt{\lambda} \cos\left(\frac{1}{\sqrt{\lambda}}\right) - \frac{1}{\pi(\pi - \frac{1}{\sqrt{\lambda}})} \sin\left(\pi - \frac{1}{\sqrt{\lambda}}\right) + \frac{1}{\pi(\pi + \frac{1}{\sqrt{\lambda}})} \sin\left(\pi + \frac{1}{\sqrt{\lambda}}\right) \right) a \\ &- \left(\lambda\pi + \frac{1}{\pi} \right) c = 0. \end{aligned}$$

Note that the eigenfunction e is identically zero if and only if $a = c = 0$. Indeed, the functions $\sin\left(\frac{t}{\sqrt{\lambda}}\right)$, $t \in [0, 1]$, and $\sin(\pi t)$, $t \in [0, 1]$, are linearly independent provided that $\lambda \neq \frac{1}{\pi^2}$, as we check below. On the contrary let us suppose that they are linearly dependent, i.e., there exist constants $a, c \in \mathbb{R}$ such that $a^2 + c^2 \neq 0$ and $a \sin\left(\frac{t}{\sqrt{\lambda}}\right) + c \sin(\pi t) = 0$, $t \in [0, 1]$. Without loss of generality, one can assume that $a \neq 0$. Then $\sin\left(\frac{t}{\sqrt{\lambda}}\right) = -\frac{c}{a} \sin(\pi t)$, $t \in [0, 1]$, and, by differentiating twice, we have $\sin\left(\frac{t}{\sqrt{\lambda}}\right) = -\frac{c}{a} \lambda \pi^2 \sin(\pi t)$, $t \in [0, 1]$. Hence $\frac{c}{a} (1 - \lambda \pi^2) \sin(\pi t) = 0$, $t \in [0, 1]$. Since $\lambda \neq \frac{1}{\pi^2}$, we have $c = 0$ yielding $\sin\left(\frac{t}{\sqrt{\lambda}}\right) = 0$, $t \in [0, 1]$, which is a contradiction. The system (2.23) and (2.24) has a nontrivial solution $(a, c) \neq (0, 0)$ if and only if its determinant is zero, which yields

$$\pi \lambda^{3/2} \cos\left(\frac{1}{\sqrt{\lambda}}\right) + \frac{1}{\pi^2(\pi - \frac{1}{\sqrt{\lambda}})} \sin\left(\pi - \frac{1}{\sqrt{\lambda}}\right) - \frac{1}{\pi^2(\pi + \frac{1}{\sqrt{\lambda}})} \sin\left(\pi + \frac{1}{\sqrt{\lambda}}\right) = 0,$$

which is equivalent to (1.10), since

$$\begin{aligned} \frac{1}{\pi^2(\pi - \frac{1}{\sqrt{\lambda}})} \sin\left(\pi - \frac{1}{\sqrt{\lambda}}\right) - \frac{1}{\pi^2(\pi + \frac{1}{\sqrt{\lambda}})} \sin\left(\pi + \frac{1}{\sqrt{\lambda}}\right) &= \frac{1}{\pi^2} \left(\frac{\sin\left(\frac{1}{\sqrt{\lambda}}\right)}{\pi - \frac{1}{\sqrt{\lambda}}} + \frac{\sin\left(\frac{1}{\sqrt{\lambda}}\right)}{\pi + \frac{1}{\sqrt{\lambda}}} \right) \\ &= \frac{2}{\pi(\pi^2 - \frac{1}{\lambda})} \sin\left(\frac{1}{\sqrt{\lambda}}\right). \end{aligned}$$

Finally, we check (1.11). By (2.22), using $b = d = 0$ and $\lambda \neq \frac{1}{\pi^2}$, we have

$$(1 - \lambda\pi^2)c = 2 \int_0^1 \sin(\pi s)e(s) ds = a \left(\frac{\sin\left(\pi - \frac{1}{\sqrt{\lambda}}\right)}{\pi - \frac{1}{\sqrt{\lambda}}} - \frac{\sin\left(\pi + \frac{1}{\sqrt{\lambda}}\right)}{\pi + \frac{1}{\sqrt{\lambda}}} \right) + c,$$

and consequently

$$c = \frac{a}{\lambda\pi^2} \left(\frac{\sin\left(\pi + \frac{1}{\sqrt{\lambda}}\right)}{\pi + \frac{1}{\sqrt{\lambda}}} - \frac{\sin\left(\pi - \frac{1}{\sqrt{\lambda}}\right)}{\pi - \frac{1}{\sqrt{\lambda}}} \right) = -\frac{2a}{\lambda\pi\left(\pi^2 - \frac{1}{\lambda}\right)} \sin\left(\frac{1}{\sqrt{\lambda}}\right),$$

which yields

$$e(t) = a \left[\sin\left(\frac{t}{\sqrt{\lambda}}\right) - \frac{2}{\lambda\pi\left(\pi^2 - \frac{1}{\lambda}\right)} \sin\left(\frac{1}{\sqrt{\lambda}}\right) \sin(\pi t) \right], \quad t \in [0, 1].$$

Using $-\frac{2}{\lambda\pi\left(\pi^2 - \frac{1}{\lambda}\right)} \sin\left(\frac{1}{\sqrt{\lambda}}\right) = \sqrt{\lambda}\pi \cos\left(\frac{1}{\sqrt{\lambda}}\right)$, one can finish the proof as in the first proof of Corollary 1.6. \square

Proof of Corollary 1.8. One can apply Theorem 1.3 with the function $g : [0, 1] \rightarrow \mathbb{R}$, $g(t) := t$, $t \in [0, 1]$. In the proof of Theorem 1.3 we have already checked that the equation $\sin\left(\frac{1}{\sqrt{\lambda}}\right) = 0$ is nothing else but the equation (1.7) in the special case $g(t) = t$, $t \in [0, 1]$. Further, taking into account $\lambda = \frac{1}{(k\pi)^2}$, $k \in \mathbb{N}$, by partial integration, we obtain that the normed eigenfunctions (1.8) take the form

$$\begin{aligned} e(t) &= C \left[\sqrt{\lambda} \cos\left(\frac{1}{\sqrt{\lambda}}\right) t + \left(\sqrt{\lambda} \sin\left(\frac{1}{\sqrt{\lambda}}\right) + \lambda \left(\cos\left(\frac{1}{\sqrt{\lambda}}\right) - 1 \right) \right) \cos\left(\frac{1}{\sqrt{\lambda}}\right) \sin\left(\frac{t}{\sqrt{\lambda}}\right) \right. \\ &\quad + \left(-\sqrt{\lambda} \cos\left(\frac{1}{\sqrt{\lambda}}\right) + \lambda \sin\left(\frac{1}{\sqrt{\lambda}}\right) \right) \sin\left(\frac{1}{\sqrt{\lambda}}\right) \sin\left(\frac{t}{\sqrt{\lambda}}\right) \\ &\quad + \cos\left(\frac{1}{\sqrt{\lambda}}\right) \cos\left(\frac{t}{\sqrt{\lambda}}\right) \int_0^t u \sin\left(\frac{u}{\sqrt{\lambda}}\right) du \\ &\quad \left. - \cos\left(\frac{1}{\sqrt{\lambda}}\right) \sin\left(\frac{t}{\sqrt{\lambda}}\right) \int_0^t u \cos\left(\frac{u}{\sqrt{\lambda}}\right) du \right] \\ &= C(-1)^k \left[\frac{1}{k\pi} t + \frac{(-1)^k - 1}{(k\pi)^2} \sin(k\pi t) + \cos(k\pi t) \int_0^t u \sin(k\pi u) du - \sin(k\pi t) \int_0^t u \cos(k\pi u) du \right] \\ &= \frac{C}{(k\pi)^2} \sin(k\pi t), \quad t \in [0, 1], \end{aligned}$$

where $C \in \mathbb{R}$ is such that $\int_0^1 (e(t))^2 dt = 1$. Since $\int_0^1 \sin^2(k\pi t) dt = \frac{1}{2}$, $k \in \mathbb{N}$, we have $C = \pm\sqrt{2}(k\pi)^2$, yielding $e(t) = \pm\sqrt{2} \sin(k\pi t)$, $t \in [0, 1]$, i.e., we have (1.13). \square

Appendix

A Connections with the paper [16] of Nazarov

In this appendix we compare the Gauss process given in (1.1) with the Gauss process given in (1.3) in Nazarov [16], and then we also compare our Theorem 1.3 with the results in Section 3 in Nazarov [16] for the KL expansions of the Gauss processes in question.

Let $(X_t)_{t \in [0,1]}$ be a zero-mean Gauss process, and suppose that its covariance function G_X is continuous. Let $\varphi : [0,1] \rightarrow \mathbb{R}$ be a measurable function such that $\int_0^1 |\varphi(s)| ds < \infty$. Introduce the function $\psi : [0,1] \rightarrow \mathbb{R}$, $\psi(t) = \int_0^1 G_X(t, s) \varphi(s) ds$, $t \in [0,1]$. Further, suppose that ψ is not the identically zero function, and let

$$q := \int_0^1 \psi(t) \varphi(t) dt = \int_0^1 \int_0^1 G_X(t, s) \varphi(t) \varphi(s) dt ds < \infty.$$

For all $\alpha \in \mathbb{R}$, introduce the stochastic process

$$(A.1) \quad X_t^{\varphi, \alpha} := X_t - \alpha \psi(t) \int_0^1 X_s \varphi(s) ds, \quad t \in [0,1].$$

Due to Nazarov [16, formula (1.4)], $(X_t^{\varphi, \alpha})_{t \in [0,1]}$ is a zero-mean Gauss process with covariance function

$$(A.2) \quad G_{X^{\varphi, \alpha}}(t, s) = G_X(t, s) + (q\alpha^2 - 2\alpha)\psi(t)\psi(s), \quad t, s \in [0,1].$$

Nazarov [16, Section 3] gave a procedure for finding the KL expansion of $(X_t^{\varphi, \alpha})_{t \in [0,1]}$ supposing that we know the KL expansion of $(X_t)_{t \in [0,1]}$. He also provided several examples (specifying X and φ), where he made the KL expansion of $(X_t^{\varphi, \alpha})_{t \in [0,1]}$ as explicit as possible. The process $X^{\varphi, \alpha}$ can be considered as a one-dimensional linear perturbation of the Gauss process X .

Now we consider two special cases of the above construction.

Let $(X_t)_{t \in [0,1]}$ be a standard Wiener process, $\varphi : [0,1] \rightarrow \mathbb{R}$ be a twice continuously differentiable function such that $q \in (0,1]$ and $\int_0^1 \left(\int_t^1 \varphi(s) ds \right)^2 dt = 1$, and let $\alpha := \frac{1 \pm \sqrt{1-q}}{q}$. Then $G_X(t, s) = t \wedge s$, $t, s \in [0,1]$, $\psi : [0,1] \rightarrow \mathbb{R}$,

$$\psi(t) = \int_0^1 (t \wedge s) \varphi(s) ds = \int_0^t s \varphi(s) ds + t \int_t^1 \varphi(s) ds, \quad t \in [0,1],$$

and $q\alpha^2 - 2\alpha = -1$. Further, $\psi'(t) = \int_t^1 \varphi(s) ds$, $t \in [0,1]$, and $\psi''(t) = -\varphi(t)$, $t \in [0,1]$, satisfying $\psi(0) = 0$, $\psi'(1) = 0$, and $\int_0^1 (\psi'(s))^2 ds = 1$. Hence, taking into account Proposition 1.1 and (A.2), choosing g as the function ψ , the Gauss process $(Y_t)_{t \in [0,1]}$ given in (1.1) coincides in law with the Gauss process $(X_t^{\varphi, \alpha})_{t \in [0,1]}$ given in (A.1).

Let $(X_t)_{t \in [0,1]}$ be a standard Wiener process, and $g : [0,1] \rightarrow \mathbb{R}$ be a twice continuously differentiable function with $g(0) = 0$, $g'(1) = 0$, and $\int_0^1 (g'(u))^2 du = 1$. Let us define the Gauss process $(X_t^{\varphi, \alpha})_{t \in [0,1]}$ given in (A.1) with $\varphi := -g''$ and $\alpha := 1$. Since

$$\int_0^1 (t \wedge s)(-g''(s)) ds = g(t), \quad t \in [0,1],$$

and

$$\int_0^1 \int_0^1 (t \wedge s)(-g''(t))(-g''(s)) dt ds = 1, \quad t \in [0,1],$$

we have $\psi = g$, $q = 1$ and $q\alpha^2 - 2\alpha = -1$. Note that it is a so-called critical case, since $q = \frac{1}{\alpha}$, see Nazarov [16, Corollary 2]. Hence the Gauss process $(X_t^{\varphi, \alpha})_{t \in [0,1]}$ given in (A.1) coincides in law with the Gauss process $(Y_t)_{t \in [0,1]}$ given in (1.1).

Based on the above discussion, if $g : [0, 1] \rightarrow \mathbb{R}$ is a twice continuously differentiable function with $g(0) = 0$, $g'(1) = 0$, and $\int_0^1 (g'(u))^2 du = 1$, then the Gauss process $(Y_t)_{t \in [0,1]}$ given in (1.1) coincides in law with one of the Gauss processes introduced in Nazarov [16, formula (1.3)].

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References

- [1] R. J. ADLER: *An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes*, Lecture Notes–Monograph Series, **12**, Institute of Mathematical Statistics, Hayward, CA, 1990.
- [2] X. AI, W. V. LI, G. LIU: Karhunen–Loève expansions for the detrended Brownian motion, *Statistics and Probability Letters* 82 (1) (2012), 1235–1241.
- [3] R. B. ASH, M. F. GARDNER: *Topics in Stochastic Processes*. Academic Press, New York, 1975.
- [4] M. BARCZY, E. IGLÓI: Karhunen–Loève expansions of alpha-Wiener bridges, *Central European Journal of Mathematics* 9 (1) (2011), 65–84.
- [5] M. BARCZY, P. KERN: Representations of multidimensional linear process bridges, *Random Operators and Stochastic Equations* 21 (2) (2013), 159–189.
- [6] H. BAUER: *Probability Theory*. de Gruyter, Berlin, 1996.
- [7] S. CORLAY, G. PAGÈS: Functional quantization-based stratified sampling methods, *Monte Carlo Methods and Applications* 21 (1) (2015), 1–32.
- [8] P. DEHEUVELS: Karhunen–Loève expansions of mean-centered Wiener processes. *High Dimensional Probability. IMS Lecture Notes–Monograph Series*, 51 (2006), 62–76.
- [9] P. DEHEUVELS, G. MARTYNOV: Karhunen–Loève expansions for weighted Wiener processes and Brownian bridges via Bessel functions, *Progress in Probability* 55 (2003), 57–93.
- [10] V. K. JANDHYALA, I. B. MACNEILL: Residual partial sum limit process for regression models with applications to detecting parameter changes at unknown times. *Stochastic Processes and their Applications* 33 (2) (1989), 309–323.

- [11] M. KAC, J. KIEFER, J. WOLFOWITZ: On tests of normality and other tests of goodness of fit based on distance methods, *The Annals of Mathematical Statistics* 26 (2) (1955), 189–211.
- [12] M. A. LIFSHITS: Bibliography of small deviation probabilities, 2015. Available at: www.proba.jussieu.fr/pageperso/smalldev/biblio.pdf
- [13] J. V. LIU: Karhunen–Loève expansion for additive Brownian motions, *Stochastic Processes and their Applications* 123 (2013), 4090–4110.
- [14] J. V. LIU, Z. HUANG, H. MAO: Karhunen–Loève expansion for additive Slepian processes, *Statistics and Probability Letters* 90 (2014), 93–99.
- [15] A. I. NAZAROV: Exact L_2 -small ball asymptotics of Gaussian processes and the spectrum of boundary-value problems, *Journal of Theoretical Probability* 22 (3) (2009), 640–655.
- [16] A. I. NAZAROV: On a set of transformations of Gaussian random functions, *Theory of Probability and Its Applications* 54 (2) (2010), 203–216.
- [17] A. I. NAZAROV, YA. YU. NIKITIN: Exact L_2 -small ball behavior of integrated Gaussian processes and spectral asymptotics of boundary value problems, *Probability Theory and Related Fields* 129 (4) (2004), 469–494.
- [18] A. I. NAZAROV, Y. P. PETROVA: The small ball asymptotics in Hilbertian norm for the Kac-Kiefer-Wolfowitz processes (in Russian), *Teoriya Veroyatnostei i ee Primeneniya* 60 (3) (2015), 482–505.
- [19] A. PAPOULIS: *Probability, Random Variables and Stochastic Processes*. McGraw-Hill Inc., New York, 1991.
- [20] J.-R. PYCKE: Une généralisation du développement de Karhunen–Loève du pont brownien, *Comptes Rendus de l'Académie des Sciences - Series I - Mathematics* 333 (7) (2001), 685–688.
- [21] J.-R. PYCKE: Multivariate extensions of the Anderson–Darling process, *Statistics & Probability Letters* 63 (4) (2003), 387–399.
- [22] J.-R. PYCKE: Un lien entre le développement de Karhunen–Loève de certains processus gaussiens et le laplacien dans des espaces de Riemann, *Ph.D. Thesis, University of Paris 6*, 2003.
- [23] I. C. TSANTILI, D. T. HRISTOPULOS: Karhunen–Loève expansion of Spartan spatial random fields, *Probabilistic Engineering Mechanics* 43 (2016), 132–147.